

# THE MAXIMAL VELOCITY OF A PHOTON

AVY SOFFER

ABSTRACT.

We estimate the probability of a photon to move faster than light, under the Einstein dynamics which, unlike the wave equation or Maxwell wave dynamics, has singular dispersion relation at zero momentum. We show that this probability goes to zero with time, using propagation estimates suitably multiscaled to control the contribution of low frequency photons.

## SECTION 0. INTRODUCTION

Let  $F(\lambda > c)$  stand for a smoothed out characteristic function of the interval  $\lambda > c$ . We say that a quantum system, with a configuration coordinates  $x$ , obeys a maximal velocity bound, if for its any evolution  $\psi(x, t)$ , localized in a compact energy interval, there is a positive constant  $c$  s.t.

$$\|F\left(\frac{|x|}{t} > c\right) \psi(x, t)\|_{L^2} \rightarrow 0,$$

as  $|t| \rightarrow \infty$ . It is fairly easy to prove such bounds for non-relativistic  $N$ -particle quantum systems (see [SS, HSS]). However, such bounds are still open for photons interacting with such systems. The goal of the present paper is to prove such bounds for a model of a single photon with an effective interaction described by a potential  $V(x)$ . This model is given by the time-dependent Schrödinger equation

$$(0.1a) \quad i \frac{\partial \psi}{\partial t} = H \psi \quad \psi|_{t=0} = \psi_0,$$

with the Schrödinger operator  $H$  on  $L^2(\mathbb{R}^3)$ , given by

$$(0.1b) \quad H := (-\Delta)^{1/2} + V(x).$$

Typeset by  $\mathcal{A}\mathcal{M}\mathcal{S}$ -TEX

This Hamiltonian is a simplified, scalar field version of the restriction of the photon field in QED, to the one particle sector. We think of the equation (0.1) as a model describing dynamics of a single photon. Moreover, it is used as a laboratory for developing the methods to study propagation with singular dispersion at zero energy, needed in tackling the full non-relativistic QED problem. Notice that the equation, without the interaction term, is relativistic invariant. Moreover, by multiplying the equation by  $-i\partial_t - |p|$ , we get the wave equation. However, since the initial data for the derived wave equation, is not localized:

$$\dot{\psi}(0) = |p|\psi(0),$$

we can not use the finite propagation property of the wave equation for this model.

We make the following assumptions:

(i)  $V(x)$ ,  $x \cdot \nabla V(x)$  are sufficiently regular and decay faster than  $O(|x|^{-2})$  at infinity;

(ii)  $H$  has no zero energy resonances or zero energy eigenvalues.

The main result of this paper is the following theorem:

**Theorem.** *Let  $\psi(x, t)$  be the solution of the Schrödinger equation (0.1), and assume, furthermore, that  $H$  satisfies the conditions (i) and (ii). If the initial data is localized in energy in some compact interval, and is such that  $\langle x \rangle^{1+\epsilon} \psi_0$  has a bounded  $L^2$  norm, we have the following maximal velocity bound:*

$$(0.2) \quad \left\| F\left(\frac{|x|}{t} > R\right) \psi(x, t) \right\|_{L^2} = o(1)$$

for  $R \gg 1$  and  $|t| \rightarrow \infty$ .

As was mentioned above, the proof of such a propagation estimates for general  $N$ -body Hamiltonians is quite elementary, see e.g. [Sig-Sof, HSS]. See also [BFS and cited references]. However, the fact that the one particle Hamiltonian of a massless photon is singular at zero energy, prevents a direct extension of the general theory of propagation estimates to the field, see however [BFSS, FGS].

The proof is based on the construction of propagation observables, on each (dyadic) scale of the energy. Suppose that the length of the momentum,  $p = -i\nabla$ , is localized in  $[2^{-n-1}, 2^{-n}]$ ,  $n \geq 0$ . Consider then the operator

$$F(A/Rt2^{-n} > 1)$$

defined through the spectral theorem.

$$(0.4) \quad A = -i/2(x \cdot \nabla + \nabla \cdot x),$$

and  $R > 1$ .

Then, we have the basic identity, derived from the Schrödinger equation:

$$(0.5) \quad \partial_t \langle \psi(t), F\psi(t) \rangle = \langle \psi(t), \{i[H, F] + \partial_t F\} \psi(t) \rangle.$$

Here,  $F$  stands for any self-adjoint operator, for which the differentiation can be justified, for the chosen  $\psi(t)$ . To get a useful propagation estimate, we use the idea of negative propagation observables, [Sig-Sof]: Suppose  $F$  stands for a family of time dependent operators, bounded from above (e.g., negative), and such that its Heisenberg derivative, defined above, is positive, up to integrable (in time) corrections. We will then obtain, upon integration over time, the following propagation estimate:

$$(0.6) \quad \int_1^T \|B\psi(t)\|^2 dt \leq C \|\psi(0)\|^2.$$

Here, we use that the Heisenberg derivative,  $D_H F$ , is given by a positive operator, plus integrable corrections:

$$(0.7) \quad D_H F = i[H, F] + \partial_t F = B^* B + R(t).$$

In general, we need to use phase space operators which, while not pseudo-differential, will have suitable phase space support. If we choose for  $F$ , the function of  $A$ , defined above, as our propagation observable, we need to estimate the commutator of  $H$  with this  $F$ , suitably localized in some energy shell. One can then use the following basic commutator expansion Lemma:

$$(0.8) \quad i[H, F(A)] = F'(A) i[H, A] + R([H, A], A),$$

where the remainder is given by an explicit multiple integral, involving the group generated by  $A$ , and, the double commutator above, of  $H$  with  $A$ . By the above expansion, we obtain, through symmetrization, that the above propagation observable, has Heisenberg derivative, that is smaller than:

$$(0.9) \quad -\frac{1}{t} G_n^2(A/t) + R(t, n).$$

Here,  $G_n$  stands for a bump function of  $\frac{A}{Rt^{2-n}}$  around 1. We will get a useful estimate, if we can control the remainder term,  $R(t, n)$ , by an integrable function of  $t$ , which is also well behaved as  $n$  tends to infinity. The leading term in the expression (0.9), comes from the time derivative of  $F$  term, of the Heisenberg

derivative. The first term in the Heisenberg derivative, the commutator, has an expansion, beginning with:

$$(0.10) \quad F'(A/t)i[H, A]\frac{2^n}{Rt},$$

which can be shown to be much smaller than the leading term, by using the energy localization around  $2^{-n}$ . To see that, we use that the commutator  $i[H, A]$  is bounded by  $c|p|$ , for some finite constant  $c$ , by the use of the uncertainty principle and our decay and regularity assumptions on  $V$ . Then, we need to show that energy localization implies a similar momentum localization. This can not be done using standard localization arguments, since the derivative of the localization function grows like  $2^n$ ! To this end, a completely different argument is used: One proves that under generic spectral assumptions on  $H$ ,  $HP_c(H)$ , dominates a constant times  $P_c(H)|p|P_c(H)$ .  $P_c(H)$  stands for the spectral projection of  $H$  on its continuous spectral part.

It then follows that:

$$(0.11) \quad E_n(H)|p|E_n(H) \leq c2^{-n}E_n(H)^2.$$

Finally, we need to show that the remainder  $R(t, n)$ , higher commutator terms in the expansion of the Heisenberg derivative, are integrable in  $t$ . This is the most involved step, without energy localization the formal expression for the remainder is given by a divergent integral. The way to estimate this last remainder term, is to use the fact, that the group generated by dilations moves the support of the energy localization functions. Consequently, the integrals over the group actions is limited to finite domains. The remainder term comes from the following expansion:

$$(0.12) \quad \begin{aligned} E_n i[|p|, \Phi_n] E_n &= E_n i[|p|^{1/2}, \Phi_n] |p|^{1/2} E_n + E_n |p|^{1/2} i[|p|^{1/2}, \Phi_n] E_n \\ &= E_n |p|^{1/2} F'_n(A/t) |p|^{1/2} E_n \frac{1}{Rt} + 2\Re E_n |p|^{1/2} R_2(A/t) E_n, \end{aligned}$$

with

$$\Phi_n \equiv E_{I_n} F\left(\frac{A}{tR2^{-n}} > 1\right) E_{I_n}$$

and with  $R_2(A/t)$  given by

$$(0.13) \quad \begin{aligned} R_2(A/t) &= \frac{1}{2} \int d\lambda \hat{F}_n(\lambda) e^{i\lambda A/Rt} \int_0^\lambda ds \int_0^s du \quad e^{-iuA/Rt} \quad \frac{1}{2}|p|^{1/2} \quad e^{iuA/Rt} (Rt)^{-2} \\ &= \frac{1}{4} \int d\lambda \hat{F}_n(\lambda) e^{i\lambda A/Rt} \int_0^\lambda ds \int_0^s du \quad e^{-u/Rt} |p|^{1/2} \quad (Rt)^{-2}. \end{aligned}$$

If we try to bound the expression for  $R_2$ , by taking the norm of the first term on the rhs, as it was done in past works, we lose a factor of  $2^{-n/2}$ , coming from localizing the  $|p|^{1/2}$  factor. On the other hand, we can not directly estimate the last expression on the rhs of equation (0.13), since the integrand grows exponentially, while  $\hat{F}_n$  decays slower than exponential, being the Fourier transform of a compactly supported function. To this end, we use the fact that the dilation group, generated by  $A$ , changes the support of functions of  $|p|$ , or  $H$ :

$$(0.14) \quad E_{I_n}(|p|)e^{i\lambda A}E_{I_n}(|p|) = 0,$$

for  $|\lambda| \geq \ln 2$ . Then, we use the mutual domination of  $|p|, H$ :

$$(0.15) \quad P_c(H)H \leq cP_c(H)|p|P_c(H) \leq dP_c(H)H,$$

for some positive constants  $c, d$ .

Combining equations (0.14), (0.15), we can then show that the integration on  $\lambda$  is limited to a compact domain, in equation (0.13). Collecting all of the above, we get estimates of the form:

$$(0.16) \quad E_n(H)\frac{1}{t}G_n^2(A/t)E_n(H) \in L^1(dt).$$

This estimate is then jacked up by the use of the propagation observable  $\frac{A}{t}F_n(A/t)$ , to obtain,

$$(0.17) \quad E_n(H)F_n(A/t)E_n(H)\frac{1}{t} \in L^1(dt).$$

In the next step of the proof, we estimate, using the above, the following operator:

$$E_n(H)F\left(\frac{|x|}{t} > R\right)E_n(H).$$

We write,  $F\left(\frac{|x|}{t} > R\right) = F\left(\frac{|x|}{t} > R\right)F_n(A/t) + F\left(\frac{|x|}{t} > R\right)\bar{F}_n(A/t)$ .

$$F_n + \bar{F}_n = 1.$$

The first term of the above decomposition, goes to zero, as time goes to infinity, by the above propagation estimates, on  $F_n(A/t)$ .

So, we need to show that the  $\bar{F}_n$  term also goes to zero.

This is formally true, since it consists of a product of two operators, with disjoint classical phase-space support, on the energy shell  $2^{-n}$ .

Again, the proof of this property necessitates the use of new phase space localization arguments. In the final step of the proof, we sum over all  $n$ , and in the process we lose some powers of  $2^{-n}$ . These are compensated by requiring the initial data to be localized in  $x$ , and by using that (up to 2) negative powers of  $|p|$ , are bounded, up to a constant, by positive powers of  $|x|$ .

## ACKNOWLEDGEMENTS

I wish to thank I.M. Sigal for useful discussions. Part of this work was done while the author visited the IHES, France. A. Soffer was partially supported by NSF grant number DMS-0903651.

## SECTION 1. PROPAGATION ESTIMATES

Our goal in this section is to prove the following key propagation estimate, as sketched in the introduction:

$$(1.1a) \quad \int_1^T \|F_n(A/t)E_n\psi(t)\|^2 \frac{dt}{t} \leq C\|E_n\psi(0)\|^2.$$

and with

$$(1.1b) \quad F_n(A/t) \equiv F\left(\frac{A}{Rt2^{-n}} > 1\right).$$

$E_n$  stands for the operator  $E_{I_n}(H)$ . We use propagation estimates, with the propagation observables

$$(1.2a) \quad \Phi_n \equiv E_{I_n}(H)F\left(\frac{A}{tR2^{-n}} > 1\right)E_{I_n}(H),$$

$$(1.2b) \quad \Phi_n \equiv E_{I_n}(H)\frac{A}{t}F\left(\frac{A}{tR2^{-n}} > 1\right)E_{I_n}(H),$$

where,  $n = 0, 1, \dots, \infty$ ,  $I_n$  stands for the interval  $[2^{-n-1}, 2^{-n}]$ ,  $E_{I_n}$  is the characteristic function of  $I_n$  and  $A$  is the dilation generator,

$$(1.3) \quad A \equiv \frac{1}{2}(x \cdot p + p \cdot x).$$

Note that  $\frac{|x|}{t} \geq R$  on the support of  $\Phi_n$ , and  $R \leq \frac{|x|}{t} \leq 2R$  on the support of  $E_n F' E_n$ . Furthermore,

$$(1.4) \quad |\partial_\lambda^j F\left(\frac{\lambda}{Rt2^{-n}} > 1\right)| \sim \left(\frac{2^n}{Rt}\right)^j.$$

The main propagation estimate is based on showing that:

$$\begin{aligned}
& E_{I_n} i \left[ |p| + V, F \left( \frac{A}{Rt2^{-n}} > 1 \right) \right] E_{I_n} \\
&= E_n \left\{ i |p|^{1/2} \left[ |p|^{1/2}, F \right] + i \left[ |p|^{1/2}, F \right] |p|^{1/2} + i[V, F] \right\} E_n \\
(1.5a) \quad &= E_n \left\{ |p|^{1/2} \tilde{F}^2 |p|^{1/2} + |p|^{1/2} \frac{2^{2n}}{R^2 t^2} Q_2(A/t) |p|^{1/2} \right\} E_n + O(t^{-2}),
\end{aligned}$$

where  $\tilde{F}^2 \sim \frac{2^n}{Rt} F'$  and,  $E_n Q_2(A/t) E_n$  is of order 1 for  $t \geq 2^n/R$ , and of order  $2^{-n}$ , for  $t \leq 2^n/R$ . Then, we show that

$$(1.5b) \quad E_n |p|^{1/2} \tilde{F}^2 |p|^{1/2} E_n \lesssim (c/Rt) E_n G_n^2 E_n,$$

with

$$G_n^2 = F\left(\frac{A}{Rt2^{-n}} \sim 1\right).$$

Letting  $F_n(\lambda) = F(2^n \lambda \geq 1)$  and using that  $-\frac{A}{t} F'_n \sim -R2^{-n} F'_n$ ,  $R \gg 1$ , we obtain

$$\begin{aligned}
(1.6) \quad & \int_1^T \partial_t \langle \psi_t, E_{I_n}(H) F_n\left(\frac{A}{Rt}\right) E_{I_n}(H) \psi_t \rangle dt = \int_1^T \langle \psi_t i[H, \Phi_n] \psi_t \rangle dt \\
& + 2^n R^{-1} \int_1^T \langle \psi_t, E_{I_n}(H) \left(\frac{-A}{t^2}\right) F'_n E_{I_n}(H) \psi_t \rangle dt.
\end{aligned}$$

Assuming that (1.5) holds, using (1.6) we can then prove:

**Theorem 1.1.** *Under the previous assumptions on  $H$ , and assume that (1.5) holds, we have the following propagation estimates:*

$$(1.7a) \quad \int_1^T \|F'_n(A/t) E_n \psi(t)\|^2 \frac{dt}{t} \leq c(R) \|E_n \psi(0)\|^2$$

$$\begin{aligned}
(1.7b) \quad & \int_1^T \|F_n(A/t) E_n \psi(t)\|^2 \frac{dt}{t} \\
& + \langle \psi(T), E_n \frac{A}{T} F_n(A/T) E_n \psi(T) \rangle \leq c(R) \| \langle A \rangle^{1/2} E_n \psi(0) \|^2
\end{aligned}$$

*proof.*

To prove (1.7a), we observe, that by the estimate (1.5), combined with Proposition(2.4), each power of  $|p|$  contributes a factor of  $2^{-n}$ , the  $\tilde{F}$  term is bounded by

$(c/Rt)F'$ , which is dominated by the integrand of the second term in equation (1.6), for  $R$  large enough. The  $Q_2$  term contribution to the commutator in equation (1.6), is bounded by our assumptions, following estimate (1.5):

$$\|E_n|p|^{1/2}\| \leq c2^{-n/2},$$

by Proposition (2.4), which eliminates one power of  $2^n$ ; then, integrating  $1/t^2$  over the region of  $t \geq 2^n/R$ , eliminates a power of  $2^n$ . The integral over  $t \leq 2^n/R$ , is also bounded, since  $Q_2$  is assumed to be smaller than  $c2^{-n}$ . To prove (1.7b), we use the propagation observable (1.3b). The proof is similar to the first case: The time derivative term, which contributes the second term on the rhs of (1.6), is, as before the dominant term. The  $Q_2$  term is controlled as before (the only difference is that now  $\hat{F}$  is replaced by  $\partial_\lambda \hat{F}(\lambda)$ , in the estimates on  $Q_2$ . As for the commutator term, we have now

$$(1.8) \quad i[H, \Phi_n] = E_n \frac{1}{t} i[H, A] F_n(A/t) E_n + E_n \frac{A}{t} F'_n(A/t) i[H, A] \frac{2^n}{Rt} E_n + E_n Q_2 E_n.$$

The  $Q_2$  term is estimated as before. The second term on the rhs of (1.8) is also estimated as before, since the factor  $A/t \sim 2^{-n}/R$ , on support  $F'$ . However, the first term is unbounded, so the bound for  $t \leq 2^n/R$ , Proposition (C), does not apply. Instead, we break the  $F$  to two parts, one decaying at infinity, and sharply localized in Fourier space, and one constant at infinity. The first part is estimated as before, and the second part is directly controlled by elementary size estimates, see Proposition (D). ■

There are new difficulties in completing this argument, compared with the usual case, without dyadic energy localization.

First, we need to minimize the number of powers of  $2^n$ , coming from expanding the function  $F_n$ . Then, we need to trade positive powers of the momentum(derivative operator)  $p = -i\nabla$ , for powers of  $2^{-n}$ .

Finally, to control the remainder term in the Commutator Expansion Lemma, the  $Q_2$  term, (or  $R_2$  term), we need to commute the derivative through the dilation group, which produces exponentially large factors.

The way out of these problems involves the following arguments. To limit the integrations in the remainder term  $R_2$ , we notice that, the dilation group moves the dyadic energy interval, away from its original support. Hence, for large enough value of the group parameter,  $\lambda$ , the fact that our propagation observable is localized



on the dyadic interval, from both sides, gives an extra decay, that cancels the exponential growth factor. This is shown in detail in the subsection "The term  $R_2$ ".

To get the  $2^{-n}$  factor from the momentum  $p$ , we prove some propositions about the properties of the operator  $H$ , which might be of independent interest. (see Proposition (2.4)) Specifically, we show, that in three (and higher) dimensions, if there are no zero energy resonances and eigenvalues, then  $H$  and  $|p|$  dominate each other, up to a multiplicative constant, on the continuous spectral subspace of  $H$ .

These estimates are the key to getting the right minimal powers of  $2^n$ , from the various propagation estimates and phase space localizations.

To prove the necessary estimates, to fill in the details of the above theorem, and the assumptions on  $Q_2$ , we estimate the Heisenberg derivative of the observables defined in (1.3), on the time intervals  $t \geq 2^n/R$ , and  $t \leq 2^n/R$ . This is the content of the following four Propositions A-D.

We begin by deriving and estimating the  $R_2$  term:

### The $R_2(A/t)$ term.

#### Proposition A.

For  $\Phi_n$  as in (1.3a), we have:

$$(1.9) \quad \int_{2^n/R}^T \langle E_n | p|^{1/2} Q_2 | p|^{1/2} E_n \rangle dt \leq \frac{c}{R} \|E_n \psi(0)\|^2.$$

*Proof.*

Direct application of the commutator expansion lemma gives:

$$(1.10a) \quad \begin{aligned} i \left[ |p|^{1/2}, F_n(A/Rt) \right] &= \int d\lambda \hat{F}_n(\lambda) e^{i\lambda A/Rt} \frac{1}{Rt} \int_0^\lambda e^{-isA/Rt} i \left[ A, |p|^{1/2} \right] e^{isA/Rt} ds \\ &= \frac{1}{2} F'_n(A/t) \frac{1}{Rt} |p|^{1/2} + \tilde{R}_2(A/t) |p|^{1/2} = \frac{1}{2} F'_n(A/t) \frac{1}{Rt} |p|^{1/2} + R_2(A/t), \end{aligned}$$

where we used that

$$(1.10b) \quad i \left[ A, |p|^{1/2} \right] = p \cdot \nabla_p |p|^{1/2} = \frac{1}{2} |p|^{1/2}.$$

$$R_2(A/t) = -\frac{1}{2} \int d\lambda \hat{F}_n(\lambda) e^{i\lambda A/Rt} \int_0^\lambda ds \int_0^s du \quad e^{-iuA/Rt} \frac{1}{2} |p|^{1/2} \quad e^{iuA/Rt} (Rt)^{-2}$$

$$(1.11) \quad = \frac{1}{4} \int d\lambda \hat{F}_n(\lambda) e^{i\lambda A/Rt} \int_0^\lambda ds \int_0^s du \quad e^{-u/Rt} |p|^{1/2} (Rt)^{-2}.$$

In general, this integral blows up at infinity, due to the fact that  $e^{u/Rt}$  grows exponentially fast, while  $\hat{F}_n(\lambda)$  decays faster than any polynomial, but not exponentially, since  $\hat{F}_n(\lambda)$  is the Fourier transform of a compactly supported function.

$$(1.12) \quad \begin{aligned} E_n i[|p|, \Phi_n] E_n &= E_n i[|p|^{1/2}, \Phi_n] |p|^{1/2} E_n + E_n |p|^{1/2} i[|p|^{1/2}, \Phi_n] E_n \\ &= E_n |p|^{1/2} F'_n(A/t) |p|^{1/2} E_n \frac{1}{Rt} + E_n |p|^{1/2} R_2(A/t) E_n, \end{aligned}$$

with  $R_2(A/t)$  given by (1.11).

$$(1.13) \quad \begin{aligned} E_n |p|^{1/2} R_2(A/t) E_n &= E_n |p|^{1/2} \tilde{E}_n(|p|) R_2(A/t) \tilde{E}_n(|p|) E_n \\ &\quad + E_n O(2^{-n}n) \bar{E}_n(|p|) |p|^{1/2} R_2(A/t) \tilde{E}_n(|p|) E_n \\ &\quad + E_n |p|^{1/2} \tilde{E}_n(|p|) R_2(A/t) \bar{E}_n(|p|) O(2^{-n}n) E_n \\ &\equiv J1 + J2 + J3. \end{aligned}$$

$$\bar{E}_n \equiv 1 - \tilde{E}_n.$$

In our case,  $1 - \tilde{E}_n = \tilde{E}(|p| \leq 1) - \tilde{E}_n$ .  $\tilde{E}_n$  stands for smoothed  $E_n$  function, and where we used proposition 2.2(d).

$$(1.14) \quad \begin{aligned} \int_1^T J_1 dt &= \int_1^T E_n |p|^{1/2} \tilde{E}_n(|p|) R_2(A/t) \tilde{E}_n(|p|) E_n dt \\ &= \int_1^T \frac{dt}{R^2 t^2} E_n |p|^{1/2} \tilde{E}_n(|p|) Q_2(A/t) \tilde{E}_n(|p|) |p|^{1/2} E_n \\ &= \int_1^{c2^n} \frac{dt}{R^2 t^2} E_n |p|^{1/2} \tilde{E}_n(|p|) Q_2(A/t) \tilde{E}_n(|p|) |p|^{1/2} E_n \\ &\quad + \int_{c2^n}^T \frac{dt}{R^2 t^2} E_n |p|^{1/2} \tilde{E}_n(|p|) Q_2(A/t) \tilde{E}_n(|p|) |p|^{1/2} E_n. \end{aligned}$$

If  $T \leq c2^n$ , then the second term on the r.h.s of (1.14) is zero.

### Estimating J1

Using proposition (2.2c), it follows that the  $\lambda$  integration (and therefore the other integrations) is limited to

$$(1.15) \quad |\lambda| \leq Rt \ln 2.$$

Hence,

$$(1.16) \quad \begin{aligned} J1 &= \frac{c}{(Rt)^2} E_n(H) |p|^{1/2} \tilde{E}_n(|p|) \int_{|\lambda| \leq Rt \ln 2} d\lambda \hat{F}_n(\lambda) e^{i\lambda A/Rt} \\ &\times \int_0^\lambda ds \int_0^s du e^{iuA/Rt} |p|^{1/2} e^{-iuA/Rt} \tilde{E}_n(|p|) E_n(H). \end{aligned}$$

$$\begin{aligned} J1 &= \frac{c}{(Rt)^2} E_n |p|^{1/2} \tilde{E}_n(|p|) \int_{|\lambda| \leq Rt \ln 2} \hat{F}_n(\lambda) e^{i\lambda A/Rt} \int_0^\lambda ds \int_0^s e^{-u/Rt} du |p|^{1/2} \tilde{E}_n(|p|) E_n(H) \\ &= \frac{1}{(Rt)^2} O(E_n 2^{-n/2} \int_{|\lambda| \leq Rt \ln 2} |\lambda^2 \hat{F}_n(\lambda)| d\lambda \quad 2^{-n/2} E_n) \\ &= \frac{1}{(Rt)^2} O(E_n 2^{-n} (Rt)^2 E_n). \end{aligned}$$

Hence,

$$\int_1^{2^n/R} J1 \quad dt \leq O\left(\frac{1}{R}\right).$$

If  $Rt > 2^n$ , we use instead, that

$$\int |\lambda^2 \hat{F}_n(\lambda)| d\lambda \leq 2^{2n},$$

so that,

$$J1 = \frac{1}{(Rt)^2} O(E_n 2^n E_n), \quad \text{and then,}$$

$$(1.17) \quad \int_{2^n/R}^T J1 dt = O\left(\frac{1}{R}\right).$$

### The estimates of $J2$ and $J3$ .

Consider the region  $Rt > 2^n$ . The integrand to estimate, which is  $(|p|_s \equiv e^{-iAs/Rt} |p| e^{iAs/Rt})$ , can be written as,

$$(Rt)^{-2} \hat{F}_n(\lambda) \tilde{J}(\lambda, s, u),$$

$$\tilde{J}(\lambda, s, u) \equiv E_n(H) |p|^{1/2} \tilde{E}_n(|p|) e^{i\lambda A/Rt} |p|_u^{1/2} \bar{E}_n(|p|) E_n(H),$$

and adjoint of such term.

First, we decompose the region of integration  $\lambda$  to:

$$\frac{|\lambda|}{Rt} > m \text{ and } \frac{|\lambda|}{Rt} \leq m, \quad m > \ln 2.$$

For  $\frac{|\lambda|}{Rt} > m$  we commute  $\tilde{E}_n(|p|)$  through  $e^{i\lambda A/Rt}$ , to get,

$$\tilde{J}(\lambda, s, u) = E_n(H)|p|^{1/2}\tilde{E}_n(|p|)e^{i\lambda A/Rt}|p|^{1/2}\bar{E}_{\bar{n}(\lambda)}(|p|)E_n(H),$$

for some  $\bar{n}(\lambda) \neq n$ , (since  $\frac{|\lambda|}{Rt} > \ln 2$ , by assumption), and so  $\bar{E}_{\bar{n}(\lambda)}E_n(H) = E_{\bar{n}(\lambda)}H$ . Then,

$$\begin{aligned} & \left| \int_{2^n/R}^T \frac{dt}{R^2 t^2} \int_{-\infty}^{\infty} \hat{F}_n(\lambda) \int_{s=0}^{\lambda} \int_{u=0}^s \tilde{J}(\lambda, s, u) d\lambda ds du \right| \\ & \leq \frac{1}{R^2} 2^{-n} R \sup_t \int_{\frac{|\lambda|}{t} > m} |\hat{F}_n(\lambda) \lambda^2| d\lambda \cdot 2^{-n/2} \cdot 2^{-n} \\ & \leq \frac{1}{R} 2^{-n/2} 2^{-2n} O(|m 2^n|^{-k}). \end{aligned}$$

In the last estimate, the factor  $2^{-n}R$  comes from the  $t$ -integration. The factor  $2^{-n/2}$  comes from  $|p|^{1/2}\tilde{E}_n(|p|)$ . This factor is missing in the adjoint terms. Another factor,  $2^{-n}$ , comes from applying proposition 2.2 to the product  $E_{\bar{n}(\lambda)}(|p|)E_n(H)$ .

Finally, we note that, by construction,  $\hat{F}_n(\lambda)$  is vanishing faster than any polynomial at infinity, and is approximately a constant, of order  $R2^{-n}$ , on an interval of size  $2^n/R$  around the origin. Since  $|\lambda| > mRt > m2^n$ , the  $O(|mt|^{-k})$  bound,  $\forall k$ , follows.

Next, we consider the region  $\frac{|\lambda|}{Rt} \leq m$ . Now,  $\frac{|\lambda|}{Rt} \leq m \Rightarrow \frac{|s|}{t}, \frac{|u|}{t} \leq m$ . Therefore  $e^{|s|/Rt}, e^{|u|/Rt} \leq e^m$ , so,  $|p|_s \leq e^m |p|$ . So, we pick up a factor of  $2^{-n}$  from  $|p|^{1/2}$  factors, and the integration of  $u, s, t$  gives:

$$\int_{\frac{|\lambda|}{Rt} \leq m} |\lambda^2 \hat{F}_n(\lambda)| d\lambda \leq c(2^{2n}/R^2)e^m.$$

Hence,

$$\left| \int_{2^n/R}^T \frac{dt}{R^2 t^2} \int \int \tilde{J}(\lambda, s, u) \hat{F}_n(\lambda) d\lambda ds du \right| \leq c \frac{e^m}{R^2} 2^{-n} R 2^{-n} (2^{2n}/R^2) 2^{-n},$$

where we gain an extra  $2^{-n}$  from  $E_{\bar{n}}E_n$  and  $|p|$  each.

**Large  $|p|$  or  $H$ .**

Since the momentum operator is unbounded, we need to bound it, under the integrals, in the definition of  $R_2$ , for large values of  $|p|$ , or  $H$ . To this end, we replace  $|p| \longrightarrow Hg(H \leq 10)$ , and then, we need to work with:

$$-|p| \longrightarrow i[A, Hg] \quad \text{and} \quad i[A, [A, Hg]].$$

$$\begin{aligned} -i[A, Hg] &= (|p| + x \cdot \nabla V)g - Hi[A, g(H)] \\ &= g(|p| + x \cdot \nabla V) - i[A, g(H)]H. \end{aligned}$$

This change is not difficult to handle as before.

In cases we only used the boundedness of these commutators, it is straightforward, since both  $g(H)|p|$  and  $i[A, g(H)]H$  are bounded.

In case we need to use a factor of  $|p|$  (or  $|p|^{1/2}$ ) to get extra smallness factor  $2^{-n}$ , it is done as before, since the  $g(H)$  factor does not change  $E_n(H)$ . The  $x \cdot \nabla V$  term is better, since it is bounded by  $c|p|^2$ . ■

**The region  $0 \leq t \leq 2^n/R$ .**

**Proposition B.**

For  $\Phi_n$  as in (1.3a), we have:

$$(1.18) \quad \int_1^{2^n/R} \|G_n(A/t)E_n\psi(t)\|^2 \frac{dt}{t} \leq c\|E_n\psi(0)\|^2.$$

Here,  $G_n$ , is a bump function of  $A/t$  around  $2^{-n}/R$ .

*Proof.*

Now we have,

$$\begin{aligned} \frac{d}{dt} (\psi(t), \Phi_n\psi(t)) &= (\psi(t), i[H, \Phi_n]\psi(t)) + \left( \psi(t), \frac{d\Phi_n}{dt}\psi(t) \right), \\ \left( \psi(t), \frac{d\Phi_n(t)}{dt}\psi(t) \right) &= \left( E_n\psi(t), t^{-1}F'_n(A/t) \left( -\frac{A}{t} \right) E_n\psi(t) \right) \\ &\leq -\frac{1}{t} \left( E_n\psi, 2^{-n}RF'_n(A/t)E_n\psi \right) \equiv -\frac{1}{t} \left( E_n\psi, \tilde{F}_n^2(A/t)E_n\psi \right), \end{aligned}$$

where,

$$\begin{aligned} |\tilde{F}_n^2(A/t)| &\lesssim 1 \quad \text{and} \quad \tilde{F}_n^2 \geq 0 \\ \tilde{F}_n^2(A/t) &\simeq 1 \quad \text{for} \quad \frac{A}{t} \sim R2^{-n} \\ \tilde{F}_n^2(A/t) &= 0 \quad \text{for} \quad \frac{A}{t} \approx R2^{-n}. \end{aligned}$$

$$\begin{aligned}
(\psi(t), i[H, \Phi_n]\psi(t)) &= (\psi(t), i(HE_n F_n E_n - E_n F_n E_n H)\psi(t)) \\
&= i(E_n \psi(t), (HE_n F_n E_n - E_n F_n E_n H)E_n \psi(t)) \\
&\leq 2 \cdot 2^{-n} \|E_n \psi(t)\|^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_0^{2^n/R} \left\{ \left( \psi(t), i[H, \Phi_n]\psi(t) \right) + \left( \psi(t), \frac{d\Phi_n}{dt}\psi(t) \right) \right\} dt \\
&\leq - \int_0^{2^n/R} \frac{dt}{t} \left\| \tilde{F}_n(A/t) E_n \psi(t) \right\|^2 + 2 \left\| E_n \psi(t) \right\|^2 \int_0^{2^n/R} 2^{-n} dt \\
&= - \int_0^{2^n/R} \frac{dt}{t} \left\| \tilde{F}_n(A/t) E_n \psi(t) \right\|^2 + \frac{2}{R} \left\| E_n \psi(t) \right\|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^{2^n/R} \frac{dt}{t} \left\| \tilde{F}_n(A/t) E_n \psi(t) \right\|^2 + \left( \psi(2^n/R), F_n(RA/2^n) \psi(2^n/R) \right) \\
&\leq \left( E_n \psi(0), F_n(A, t=0) E_n \psi(0) \right) + \frac{2}{R} \left\| E_n \psi(t) \right\|^2.
\end{aligned}$$

■

### Improved Decay

#### Proposition C.

Under the assumptions as the above Propositions (A,B), we have the following propagation estimate,

$$\begin{aligned}
&\int_{2^n/R}^T \|F_n \left( \frac{A}{Rt2^{-n}} > 1 \right) E_n \psi(t)\|^2 \frac{dt}{t} \\
(1.19) \quad &+ \langle \psi(T), E_n \frac{A}{T} F_n \left( \frac{A}{RT2^{-n}} > 1 \right) E_n \psi(T) \rangle \leq c(R) \|E_n \langle A \rangle^{1/2} E_n \psi(0)\|^2,
\end{aligned}$$

for all  $R$  sufficiently large.

*Proof.* We can use  $E_n \frac{A}{t} F_n(A/t) E_n = \Phi_n$ . The estimate of  $R_2(A/t)$  is the same as before, so for  $t \geq 2^n$ , it is done as before. The main change is that now  $\hat{F}(\lambda)$  is replaced by  $\partial_\lambda \hat{F}(\lambda)$ , and a similar proof applies. The first term in the commutator expansion of  $i[H, \Phi_n]$ , has an extra, positive term, which is however integrable over time by the Previous propositions (A,B), since it is supported (in phase space) on the support of  $F'_n$ . ■

For  $0 \leq t \leq 2^n$ , since  $\frac{A}{t} F_n$  is not bounded, the proof is different.

**Proposition D.**

$$(1.19) \quad \int_1^{2^n/R} \|F_n \left( \frac{A}{Rt2^{-n}} > 1 \right) E_n \psi(t)\|^2 \frac{dt}{t} \\ + \langle \psi(2^n/R), E_n \frac{A}{2^n/R} F_n (A > 1) E_n \psi(t) \rangle \leq c(R) \|E_n < A >^{1/2} E_n \psi(0)\|^2,$$

for all  $R$  sufficiently large.

*Proof.*

We write,

$$\frac{A}{t} F_n = F_{n,1} + F_{n,2},$$

where,

$$F_{n,1} = i \int e^{-\lambda^2} \partial_\lambda \hat{F}_n(\lambda) e^{-i(A/t)\lambda} d\lambda,$$

and

$$F_{n,2} \equiv (A/t) F_n - F_{n,1}.$$

Then,  $F_{n,2}$  is bounded, and the previous proof applies, while  $F_{n,1} \equiv \frac{A}{t} G_n$ , with  $G_n$  smooth, approaching a constant at infinity. Then,

$$i[H, F_{n,1}] = \frac{1}{t} i[H, A] G_n + \frac{A}{t} i[H, G_n] \\ = \frac{1}{t} [|p| + \tilde{V}(x)] G_n + \frac{A}{t} \int \hat{G}_n(\lambda) e^{-i\lambda A/t} \int_0^\lambda e^{isA/t} [H, A/t] e^{-isA/t} ds d\lambda.$$

The first term is bounded by  $2^{-n} \|G_n\|$  on support of  $E_n$ , and therefore the integral over  $1 \leq t \leq 2^n$  is bounded by  $O(1)$ . The second term is,

$$\frac{c}{t} \int \partial_\lambda (\lambda \hat{G}_n(\lambda)) e^{-i\lambda A/t} \frac{1}{\lambda} \int_0^\lambda e^{-s/t} [|p| + \tilde{V}(x, s)] ds \\ + \frac{c}{t} \int \hat{G}_n(\lambda) e^{-i\lambda A/t} \frac{1}{\lambda} \int_0^\lambda e^{-s/t} [|p| + \tilde{V}(x, s)] ds. \\ \frac{1}{\lambda} \int_0^\lambda e^{-s/t} ds = -\frac{t}{\lambda} e^{-s/t} \Big|_0^\lambda = \frac{t}{\lambda} (e^{-\lambda/t} - 1)$$

and, for  $\frac{\lambda}{t} \ll 1$ ,  $e^{-\lambda/t} - 1 \sim -\frac{\lambda}{t} + \frac{1}{2} \frac{\lambda^2}{t^2}$ ,

so the first term contributes  $\int \hat{G}_n(\lambda) e^{i\lambda A/t} d\lambda = O(1)$ , and the second term is bounded by  $c \int |\lambda \hat{G}_n(\lambda)| d\lambda / t$ .

Furthermore, there is a factor of  $2^{-n}$ , coming from  $|p| + \tilde{V}(x, s) \leq |p| + c|p|^2$ .

Here,  $\tilde{V}(x, s) = e^{isA/t} \tilde{V}(x) e^{-isA/t}$ . ■

## SECTION 2. AUXILIARY IDENTITIES AND INEQUALITIES

**Lemma 2.1.** *Assume  $H = |p| + V$  and  $|V| < \frac{1}{2r}$ . Then,*

$$(2.1) \quad \left\| |p| E_{I_n}(H) f \right\| + \left\| V E_{I_n}(H) f \right\| \leq c 2^{-n} \left\| E_{I_n} f \right\|.$$

*Proof.* For  $\|f\| = 1$ ;

$$\begin{aligned} \left( f, E_{I_n}(|p|^2 + V^2) E_{I_n} f \right) &= \left( f, E_{I_n}(H) \{H^2 - V|p| - |p|V\} E_{I_n}(H) f \right) \\ &\leq 2^{-2n} \left\| E_{I_n} f \right\|^2 + 2 \left\| V E_{I_n}(H) f \right\| \left\| |p| E_{I_n}(H) f \right\|. \end{aligned}$$

Let  $a = \left\| |p| E_{I_n} f \right\|$   $b = \left\| V E_{I_n} f \right\|$ . Then,  $b < (1 - \delta) \left\| |p| E_{I_n} f \right\|$  by the uncertainty inequality (in 3-dimensions or higher), and so,

$$\begin{aligned} \left( f, E_{I_n}(|p|^2 + V^2) E_{I_n} f \right) &= (a^2 + b^2 - 2ab) + 2ab \\ &\geq \delta^2 \left\| |p| E_{I_n} f \right\|^2 + 2ab. \end{aligned}$$

It follows that,  $\delta^2 \left\| |p| E_{I_n} f \right\|^2 \leq 2^{-2n} \left\| E_{I_n} f \right\|^2$ . ■

**Proposition 2.1.** *If  $|p| \lesssim H$ , then (2.1) holds.*

*Proof.* When  $|p| \leq mH$ , we have that, by the spectral theorem,

$$(2.2) \quad \frac{1}{H} \leq \frac{m}{|p|} \implies \left\| \frac{1}{H^{1/2}} f \right\|_{L^2} \leq m^{1/2} \left\| \frac{1}{|p|^{1/2}} f \right\|.$$

Hence,

$$\begin{aligned} V(x) \frac{1}{H} &= V(x) \left( \frac{1}{|H|} - \frac{1}{|p|} \right) + V(x) \frac{1}{|p|}. \\ \left\| V(x) \frac{1}{|p|} \right\| &= \left\| V(x) r \frac{1}{r} \frac{1}{|p|} \right\| \leq \left\| V(x) \frac{1}{r} \right\|_{\infty} 2 \left\| |p| \frac{1}{|p|} \right\| < \infty, \end{aligned}$$

$$\begin{aligned} V(x) \left( \frac{1}{H} - \frac{1}{|p|} \right) &= -V(x) \frac{1}{H} V(x) \frac{1}{|p|} \\ &= -V(x) \frac{1}{H} \frac{1}{r} r V(x) r \frac{1}{r} \frac{1}{|p|} \\ &= -V(x) r \frac{1}{r} \frac{1}{H} \frac{1}{r} r^2 V(x) \frac{1}{r} \frac{1}{|p|}. \end{aligned}$$



Since,

$$\frac{1}{r} \frac{1}{H} \frac{1}{r} = \frac{1}{r} \frac{1}{H^{1/2}} \frac{1}{H^{1/2}} \frac{1}{r} = \left( \frac{1}{H^{1/2}} \frac{1}{r} \right)^* \left( \frac{1}{H^{1/2}} \frac{1}{r} \right),$$

we get that,

$$\begin{aligned} \left\| V(x) \left( \frac{1}{H} - \frac{1}{|p|} \right) \right\| &\leq \left\| \frac{1}{H^{1/2}} \frac{1}{r} r V(x) \right\| \left\| \frac{1}{H^{1/2}} \frac{1}{r} r^2 V(x) \frac{1}{r} \frac{1}{|p|} \right\| \\ &\leq m \left\| \frac{1}{|p|^{1/2}} \frac{1}{r} (r V(x)) \right\| \left\| \frac{1}{|p|^{1/2}} \frac{1}{r} (r^2 V) \right\| \left\| \frac{1}{r} \frac{1}{|p|} \right\| \\ &< m C_V, \end{aligned}$$

where  $C_V$  is a constant, depending on the  $L^\infty$  norm of  $V, rV, r^2V$ .

We therefore conclude that,

$$|p| \frac{1}{H} = (|p| + V - V) \frac{1}{H} = H \frac{1}{H} - V \frac{1}{H} = 1 - V \frac{1}{H}$$

is also bounded. Finally, we have that,

$$|p| E_{I_n}(H) = |p| \frac{1}{H} H E_{I_n}(H) = O(1) 2^{-n} E_{I_n}(H). \quad \blacksquare$$

Now we prove some useful identities.

**Lemma 2.2.**

- (i)  $r^2(-\Delta) = A^2 + L^2 - iA - 3/4$ , (*dimension*=3).
- (ii)  $e^{i\lambda A} |p|^\alpha e^{-i\lambda A} = e^{-\alpha\lambda} |p|^\alpha$ ;  $A \equiv \frac{1}{i}(r\partial_r + \frac{3}{2})$ .
- (iii)  $e^{i\lambda A} r^\alpha e^{-i\lambda A} = e^{\alpha\lambda} r^\alpha$ .
- (iv)  $\frac{1}{|p|} A \frac{1}{r} = \frac{1}{|p|} \left( i|p| \frac{\partial}{\partial |p|} - \frac{3i}{2} \right) \frac{1}{r} = \frac{1}{|p|} \left( ip \cdot \frac{\partial}{\partial p} - \frac{3i}{2} \right) \frac{1}{r} = O(1)$ .
- (v)  $\left\| \frac{1}{|p|} \frac{1}{r} \psi \right\| \leq 2 \|\psi\|$ .

*Proof.*

- (i)  $r^2(-\Delta) = r^2 \left( -\partial_r^2 - \frac{2}{r} \partial_r + \frac{L^2}{r^2} \right) = -r^2 \partial_r^2 - 2r \partial_r + L^2$   
 $= -2r \partial_r + 2r \partial_r - r \partial_r^2 r + L^2 = -r \partial_r^2 r + L^2$ ,  
 $A^2 = -\frac{1}{4} \left( -2r \partial_r - 3 \right)^2 = (-4r \partial_r r \partial_r - 9 - 12r \partial_r) \frac{1}{4}$   
 $= r \partial_r - r \partial_r^2 r - \frac{9}{4} - 3r \partial_r = -r \partial_r^2 r - 2r \partial_r - \frac{9}{4}$ ;  
 so,  $r^2(-\Delta) = -r \partial_r^2 r + L^2 = A^2 + \frac{9}{4} + L^2 - iA - \frac{3}{4} = A^2 + L^2 - iA - \frac{3}{4}$ .
- (ii)  $\partial_\lambda \left( e^{i\lambda A} |p|^\alpha e^{-i\lambda A} \right) = e^{i\lambda A} i \left[ A, |p|^\alpha \right] e^{-i\lambda A} = -\alpha e^{i\lambda A} |p|^\alpha e^{-i\lambda A}$   
 $\Rightarrow e^{i\lambda A} |p|^\alpha e^{-i\lambda A} = e^{-\lambda\alpha} |p|^\alpha$ .
- (iii) As in(ii), but now,  $i[A, r^\alpha] = \alpha r^\alpha$ .
- (iv) Follows from (v).
- (v)  $\left\| \frac{1}{|p|} \frac{1}{r} \psi \right\|_{L^2} \leq 2 \left\| r \frac{1}{r} \psi \right\|_{L^2} = 2 \|\psi\|$  by the uncertainty inequality in 3 dimensions.  $\blacksquare$

**Proposition 2.2.**

$$a) \quad E_n(H) = E_n(H)E_n(|p|)E_n(H) + E_n(H)\delta^{-1}O(2^{-n})E_n(H).$$

$$b) \quad E_{\bar{n}}(|p|)E_n(H) = E_{\bar{n}}(|p|)O(2^{-n})E_n(H), \quad \bar{n} \neq n.$$

For  $\eta > \ln 2$ ,

$$c) \quad E_n(|p|)e^{i\eta A}E_n(H) = E_n(|p|)O(2^{-n})E_n(H).$$

$$d) \quad \bar{E}_n(|p|)E_n(H) = \bar{E}_n(|p|)O(2^{-n})E_n(H).$$

*Proof.* Part(a).

$$\begin{aligned} E_n(H) &= -E_n(E_n(|p|) - E_n)E_n + E_n(H)E_n(|p|)E_n(H), \\ E_n(|p|) - E_n(H) &= \int \hat{E}_n(\lambda) (e^{i\lambda|p|} - e^{i\lambda H}) d\lambda \\ &= \int \hat{E}_n(\lambda) e^{i\lambda H} \int_0^\lambda e^{-isH} (-i)V e^{is|p|} ds d\lambda. \end{aligned}$$

$$|V| \leq c < x >^{-2},$$

$$\begin{aligned} \Rightarrow & |(\phi, E_n[E_n(|p|) - E_n(H)] E_n \psi)| \\ & \leq c \int \| < x >^{-1} e^{-isH} e^{i\lambda H} E_n \phi \| |\hat{E}_n(\lambda) \lambda| \frac{1}{\lambda} \int_0^\lambda \| < x >^{-1} e^{-is|p|} E_n \psi \| ds d\lambda \\ & \leq c\delta^{-1} \|HE_n \phi\| \int |\lambda \hat{E}_n(\lambda)| d\lambda \| |p| E_n \psi \| \\ & \leq c\delta^{-2} 2^{-n} 2^n 2^{-n} \|\phi\| \|\psi\| = c\delta^{-2} 2^{-n} \|\phi\| \|\psi\|, \end{aligned}$$

where we used that,

$$< x >^{-1} \leq c|p| \leq c\delta^{-1}H$$

in dimension three or higher, and proposition(2.1).

Notice that,  $\|pE_n \psi\| \leq c\delta^{-1} \|HE_n \psi\| \leq c\delta^{-1} 2^{-n} \|\psi\|$ , by Lemma 2.2.

Part(b).

$$\begin{aligned} |(\phi, E_{\bar{n}}(|p|)E_n(H)\psi)| &= |(\phi, E_{\bar{n}}(|p|)[E_{\bar{n}}(|p|) - E_{\bar{n}}(H)]E_n(H)\psi)| \\ &\leq c\delta^{-1} \|HE_n \psi\| \| |p| E_{\bar{n}}(|p|) \phi \| \int |\hat{E}_{\bar{n}} \lambda| d\lambda, \end{aligned}$$

by the proof of Part(a). The last expression is therefore bounded by,

$$c\delta^{-1} 2^{-n} \|\psi\| 2^{-\bar{n}} \|\phi\| 2^{\bar{n}} = c\delta^{-1} 2^{-n} \|\phi\| \|\psi\| = O(2^{-n}).$$

Part(c) follows from Part(b), since, for  $|\lambda| > \ln 2$ :

$$\begin{aligned} E_n(|p|)e^{i\lambda A} &= E_n(|p|)e^{i\lambda A}E_n(e^{-\lambda}|p|) \\ &\lesssim E_n(|p|)e^{i\lambda A} \sum_{|\bar{n}| \leq M} E_{\bar{n}}(|p|), \end{aligned}$$

with  $\bar{n} \neq n, M < \infty$ .

Part(d) follows from part(b), since the domain of  $|p|$ , in the support of  $\bar{E}_n(|p|)$ , is covered by  $n$  dyadic intervals from  $[2^{-n}, 1]$ . ■

**Proposition 2.3.** *Assume that  $H \equiv |p| + V(x)$  has no bound states or zero energy resonances, and that  $V(x)$  vanishes faster than  $r^{-2}$  at infinity, and is sufficiently regular. The dimension is 3. Then, for some  $0 < m < \infty$ ,*

$$(2.3) \quad H \geq m|p|.$$

*Proof.* Since  $H$  has no bound states, and  $V(x) \rightarrow 0$  at infinity,  $H \geq 0$ . If we now make a small perturbation,  $H \rightarrow H_\epsilon = H + \epsilon V$ , then, since  $H$  has no zero energy resonances,  $H_\epsilon$  has no bound states, for  $\epsilon$  sufficiently small. Hence  $H_\epsilon \geq 0$ .

But then,

$$H = \frac{1}{1+\epsilon}(|p| + (1+\epsilon)V) + \left(1 - \frac{1}{1+\epsilon}\right)|p| = \frac{1}{1+\epsilon}H_\epsilon + m|p| \geq m|p|. \blacksquare$$

**Lemma 2.3.** *For  $H = |p| + V$ , we have that  $|p| \geq \delta H$ , for dimension=3, provided,  
 $|V| \leq \bar{c}/r$ .*

*Proof.* For  $\delta > 0$ ,

$$(2.4) \quad \delta H = \delta|p| + \delta V \leq \delta|p| + \delta c|p| = \delta(1+c)|p| \leq |p|,$$

where we used that for  $|V| \leq \bar{c}|x|^{-1}$ , we have  $|V| \leq c|p|$  in 3 dimensions. So, in particular, we have,

$$(2.5) \quad E_{I_n}(H)|p|E_{I_n}(H) \geq E_{I_n}(H)\delta H E_{I_n}(H) \geq \delta \inf I_n E_{I_n}(H).$$

■

**Proposition 2.4.** *Suppose, as before, that  $H \equiv |p| + V(x)$ , the dimension is 3, and that  $V(x)$  is sufficiently regular, and vanishes faster than  $r^{-2}$  at infinity. Suppose, moreover, that  $H$  has no zero energy resonances, and no zero energy bound states. Then,*

$$(2.6) \quad P_c(H)HP_c(H) \geq P_c(H)\delta|p|P_c(H), \quad \text{for some } \delta > 0.$$

*Proof.* We have that,

$$P_c(H)(|p| + V)P_c(H) \geq 0.$$

Add a small perturbation  $\epsilon V$  to  $H$ :  $H_\epsilon \equiv H + \epsilon V = |p| + (1+\epsilon)V$ . Then, for  $\epsilon$  sufficiently small, no new bound states are created. Hence,

$$(2.7) \quad f(H_\epsilon \geq -\epsilon_0)(|p| + V + \epsilon V)f(H_\epsilon \geq -\epsilon_0) \geq 0,$$

for some  $\epsilon_0 > 0$ , and  $f$  is a smooth characteristic function of the interval  $[-\epsilon_0, \infty]$ . Then,

$$P_c(H)HP_c(H) = P_c(H)\left(\frac{1}{1+\epsilon}H_\epsilon + \epsilon_1|p|\right)P_c(H),$$

would imply that:

$$P_c(H)HP_c(H) \geq \epsilon_1 P_c(H)|p|P_c(H),$$

if we can prove that for some  $\epsilon_2 < \epsilon_1$ ,

$$(2.8) \quad P_c(H)H_\epsilon P_c(H) \geq P_c(-\epsilon_2|p|)P_c.$$

So, want to use (2.7) to prove (2.8).

$$(2.9) \quad \begin{aligned} P_c(H)H_\epsilon P_c(H) &= P_c(H)f(H_\epsilon \leq -\epsilon_0)H_\epsilon P_c(H) + P_c(H)f(H_\epsilon \geq -\epsilon_0)H_\epsilon P_c(H) \\ &\geq P_c(H)f(H_\epsilon \leq -\epsilon_0)H_\epsilon P_c(H), \end{aligned}$$

by (2.7). Now,

$$\begin{aligned} P_c(H)f(H_\epsilon \leq -\epsilon_0) &= P_c(H)\left[f(H_\epsilon \leq -\epsilon_0) - f(H \leq -\epsilon_0)\right] \\ &= P_c(H) \int \hat{f}(\lambda)e^{+i\lambda H}i \int_0^\lambda e^{-isH}\epsilon V e^{isH_\epsilon}ds \\ &= P_c(H)O(\epsilon V) \int |\lambda \hat{f}(\lambda)|d\lambda \\ &= P_c(H)O(\epsilon V)O(\epsilon_0^{-1}), \end{aligned}$$

so,

$$\begin{aligned} P_c(H)f(H_\epsilon \leq -\epsilon_0)H_\epsilon P_c(H) &= P_c(H)O(\epsilon V/\epsilon_0)\tilde{f}(H_\epsilon \leq -\epsilon_0)H_\epsilon \left(f(H_\epsilon \leq -\epsilon_0) + \bar{f}(H_\epsilon \leq -\epsilon_0)\right)P_c(H) \\ &= P_c(H)O(\epsilon V/\epsilon_0)\tilde{f}(H_\epsilon)H_\epsilon f(H_\epsilon \leq -\epsilon_0)P_c(H) \\ &= P_c(H)O(\epsilon V/\epsilon_0)\tilde{f}(H_\epsilon)H_\epsilon O(\epsilon V/\epsilon_0)P_c, \end{aligned}$$

since  $\tilde{f}(H_\epsilon \leq -\epsilon_0)\bar{f}(H_\epsilon \leq -\epsilon_0) = 0$ . ( $f + \bar{f} \equiv 1$ ,  $\tilde{f}f = f$ ). The last term, can be bounded by:

$$(2.11) \quad P_c(H)\epsilon^2 \langle x \rangle^{-2} O_{\epsilon_0}(H_\epsilon) \langle x \rangle^{-2} P_c(H) \leq O_{\epsilon_0}(1)P_c\epsilon^2|p|P_c,$$

$$O_{\epsilon_0} \sim O(1/\epsilon_0^2)H_\epsilon, \quad \text{coming from :}$$

$$\int |\lambda \hat{f}(\lambda)|d\lambda = O(1/\epsilon_0).$$

$\epsilon_0$  is basically the distance of zero to the largest (negative) e.v. of  $H_\epsilon$ , and  $\epsilon$  is arbitrarily small, so  $\epsilon^2/\epsilon_0^2 \leq O(\epsilon^2)$ . Hence, (2.9) - (2.11) imply (2.8). ■

## SECTION 3. MAXIMAL VELOCITY BOUND

We begin with estimating, for  $a > 1$ ,

$$(3.1) \quad E_n(H)F_a\left(\frac{|x|}{t} > a\right)E_n(H) \equiv E_nF_aE_n.$$

$$(3.2) \quad E_nF_aE_n = E_nF_a(F_n(A/t) + \bar{F}_n(A/t))E_n,$$

where,

$$(3.3) \quad F_n(A/t) \equiv F_n\left(\frac{A}{t} > R2^{-n}\right) = 1 - \bar{F}_n,$$

$$(3.4) \quad 1 < R < a,$$

and such that

$$(3.5) \quad \bar{F}_n(2^{-n}b)F_a(b) = 0.$$

Since, by the propagation estimates,

$$\|F_n(A/t)E_n(H)\psi(t)\| \leq o(1)\|E_n\psi_0\|,$$

with  $o(1) \rightarrow 0$  as  $t \rightarrow \infty$ , we need to control  $E_nF_a\bar{F}_nE_n$  by a decaying function of  $t$ , in order to prove the maximal velocity bound, on the energy shell  $I_n$ .

**Proposition 3.1.**

$$\|E_nF_aE_n\| \leq o(1)\|E_n\psi\| + \frac{c2^n}{t}\|E_n\psi\|$$

*Proof.*

$$(3.6) \quad \begin{aligned} E_nF_a\bar{F}_nE_n &= E_n(|p|^{-1}|p|^{-1}\Delta)r^2F_ar^{-2}\bar{F}_nE_n \\ &= E_n|p|^{-1}|p|^{-1}(A^2 + cA + c_1)(r^{-2}F_a)\bar{F}_nE_n \\ &= c_2E_n|p|^{-1}(|p|^{-1}Ar^{-1})(A + c)r^{-1}F_a\bar{F}_nE_n \\ &\quad + c_3E_n|p|^{-1}(|p|^{-1}r^{-1})r^{-1}F_a\bar{F}_nE_n \equiv B_1 + B_2. \end{aligned}$$

$$(3.7) \quad B_2 = E_n O(2^n) O_0(1) c_3 \frac{1}{at} \left[ \left( \frac{ta}{r} \right) F_a \right] \bar{F}_n E_n,$$

where we used that,

$$E_n |p|^{-1} = O(2^{+n}) \quad \text{and} \quad |p|^{-1} r^{-1} = O_0(1).$$

Since  $\frac{ta}{r} F_a = O(1)$ ,  $B_2 = O(2^n/at)$ ,

$$\begin{aligned} B_1 &= E_n O(2^n) O_1(1) c \frac{1}{at} \left( \frac{ta}{r} F_a \right) \bar{F}_n E_n \\ &\quad + E_n O(2^n) O_1(1) \frac{1}{at} \left( \frac{ta}{r} F_a \right) R t (R t)^{-1} A \bar{F}_n E_n \\ &\quad + E_n O(2^n) O_1(1) \frac{1}{at} \left[ A, \left( \frac{at}{r} F_a \right) \right] \bar{F}_n E_n, \end{aligned}$$

where

$$O_1(1) \equiv |p|^{-1} (A - 3/2i) r^{-1} = |p|^{-1} p \cdot x r^{-1} = \sum_{k=1}^3 \frac{p_k}{|p|} \frac{r_k}{r}.$$

Therefore,

$$\begin{aligned} B_1 &= E_n O(2^n/at) \bar{F}_n E_n + E_n O(2^n) O_1(1) \left( \frac{ta}{r} F_a \right) \frac{R}{a} O(2^{-n}) \bar{F}_n E_n \\ (3.8) \quad &= O(2^n/at) + O\left(\frac{R}{a}\right). \end{aligned}$$

The key to this computation is the repeated use, as we do below, of the following:

$$\begin{aligned} E_n &= E_n |p|^{-1} |p|^{-1} (A^2 + cA + c_1) r^{-2} \\ &= c_2 E_n |p|^{-1} (|p|^{-1} A r^{-1}) (A + c') r^{-1} + c_3 E_n |p|^{-1} (|p|^{-1} r^{-1}) r^{-1} \\ &= E_n |p|^{-1} O_1(1) (A + c') r^{-1} + c_3 E_n |p|^{-1} (|p|^{-1} r^{-1}) r^{-1} \\ (3.9) \quad &+ c_4 E_n |p|^{-1} (|p|^{-1} r^{-1} A) r^{-1}. \end{aligned}$$

We apply it again, to the  $O\left(\frac{R}{a}\right)$  term:  $(F^{[k]}(\lambda) \equiv \lambda^k F(\lambda))$ ;

$$\begin{aligned} O(R/a) &\sim \frac{R}{a} E_n O_1(1) F_a^{[-1]} \bar{F}_n^{[1]} E_n \\ &= \frac{R}{a} E_n |p|^{-1} O_1(1) (A + c') r^{-1} O_1(1) F_a^{[-1]} \bar{F}_n^{[1]} E_n \\ (3.10) \quad &+ \frac{R}{a} E_n |p|^{-1} O_0(1) r^{-1} O_1(1) F_a^{[-1]} \bar{F}_n^{[1]} E_n. \end{aligned}$$

Now, the important observation is that,

$$r^{-1}|p|^{-1}Ar^{-1} = (r^{-1}|p|^{-1}A)r^{-1}$$

and

$$r^{-1}O_1(1) = O_1(1)r^{-1} - r^{-1}O(|p|^{-1})r^{-1} = O_1(1)r^{-1} + O(1)r^{-1}$$

$$(3.11) \quad [A, O_1(1)] \sim O_1(1).$$

We derive:

$$O_1(1)(A + c')r^{-1}O_1(1)F_a^{[-1]}\bar{F}_n^{[1]} = O_1(1)(A + c')(r^{-1}p^{-1}Ar^{-1})F_a^{[-1]}\bar{F}_n^{[1]},$$

$$\begin{aligned} Ar^{-1}F_a^{[-1]}\bar{F}_n^{[1]} &= r^{-1}F_a^{[-1]}A\bar{F}_n^{[1]} + cr^{-1}F_a^{[-1]}\bar{F}_n^{[1]} \\ &\quad + \frac{1}{at}(\partial_r F_a^{[-1]})\bar{F}_n^{[1]} \\ &= \frac{1}{at}F_a^{[-2]}(Rt)2^{-n}\bar{F}_n^{[2]} + \frac{c}{at}F_a^{[-2]}\bar{F}_n^{[1]} + \frac{1}{at}(\partial_\lambda F_a^{[-1]})\bar{F}_n^{[1]} \\ &= O\left(\frac{1}{at}\right) + O(2^{-n}\frac{R}{a}F_a^{[-2]}\bar{F}_n^{[2]}). \end{aligned}$$

$$\partial_\lambda F(\lambda = x) \equiv \left. \frac{\partial F}{\partial \lambda} \right|_{\lambda=x}.$$

Using that  $Ar^{-1}p^{-1} = O_1(1) + O_0(1)$ , we have,

$$\begin{aligned} \frac{R}{a}E_n O_1(1)F_a^{[-1]}\bar{F}_n^{[1]}E_n &= \frac{R}{a}E_n |p|^{-1}O_1(1)(r^{-1}|p|^{-1}A + c')r^{-1}F_a^{[-1]}\bar{F}_n^{[1]}E_n \\ &\quad + \frac{R}{a}E_n |p|^{-1}O_1(1)^2 r^{-1}F_a^{[-1]}A\bar{F}_n^{[1]}E_n \\ (3.12) \quad &= O\left(\frac{R}{a}\frac{2^n}{at}\right) + E_n \left(\frac{R}{a}\right)^2 O_{0,1}(1)^2 F_a^{[-2]}\bar{F}_n^{[2]}E_n. \end{aligned}$$

where we used that,

$$E_n |p|^{-1} = E_n O(2^{-n}), \quad r^2 \Delta = A^2 + cA + c',$$

$$\frac{A}{R2^{-n}t}\bar{F}_n = O(1),$$

$$|p|^{-1}Ar^{-1} = O(1).$$

We now do this computation again, this time, for the term of order  $(\frac{R}{a})^2$ :  
Doing it  $k$  times, we get:

$$(3.13) \quad \left(\frac{R}{a}\right)^k E_n O(1)^k F_a^{[-k]} \bar{F}_n^{[k]} E_n + \sum_{j=1}^k E_n O\left(\frac{2^n}{t}\right) \left(\frac{R}{a}\right)^{(j-1)} \frac{1}{a} O(1)^j \tilde{F}_a^{[-j]} \bar{F}_n^{[j-1]} E_n,$$

with,

$$(3.14) \quad \tilde{F}_a^{[-k]} = \frac{r}{at} F_a^{[-k+1]} + r \partial_r \frac{r}{at} F_a^{[-k+1]}.$$

For  $k \sim \delta \ln t$ , and  $\frac{R}{a} < 1$  sufficiently small, we have that the  $E_n F_a \bar{F}_n E_n$  term is bounded by  $O(t^{-1} 2^n)$ . ■

We can now prove the Theorem on Maximal velocity bound.

### Proof of Maximal Velocity Bound

Now,

$$(3.15) \quad \begin{aligned} & \left( \psi(t), F_a^2 \left( \frac{r}{t} > a \right) \psi(t) \right) \\ &= \sum_n \left( H^{-1/2} \psi(t), F_a^2 E_n H^{1/2} \psi(t) \right) + Q \\ &= \sum_n \left( H^{-1/2} \psi(t), F_a^2 (F_n(A/t) + \bar{F}_n(A/t)) E_n H^{1/2} \psi(t) \right) + Q \\ &= \sum_n (H^{-1/2} \psi, F_a^2 F_n E_n H^{1/2} \psi) + \sum_n (H^{-1/2} \psi, F_a^2 \bar{F}_n E_n H^{1/2} \psi) + Q. \end{aligned}$$

$$(3.16) \quad \begin{aligned} & \left| \sum_n (H^{-1/2} \psi, F_a^2 F_n E_n H^{1/2} \psi) \right| \\ & \leq \|F_a H^{-1/2} \psi\| \sum_n \langle n \rangle^{-1/2-\varepsilon/2} \langle n \rangle^{1/2+\varepsilon/2} \|F_n E_n H^{1/2} \psi\| \\ & \leq c \|F_a H^{-1/2} \psi\| \left( \sum_n \langle n \rangle^{1+\varepsilon} \|F_n E_n H^{1/2} \psi\|^2 \right)^{1/2} \\ & \leq c \|F_a H^{-1/2} \psi\| \left( \sum_n o(1) \|H^{1/2} E_n \langle x \rangle^{1/2} \psi\|^2 \langle n \rangle^{1+\varepsilon} \right)^{1/2} \\ & \leq c \|F_a H^{-1/2} \psi\| o(1) \|\langle x \rangle^{1/2} \psi\|, \end{aligned}$$



$$\begin{aligned}
\left| \sum_n (H^{-1/2}\psi, F_a^2 \bar{F}_n E_n H^{1/2}\psi) \right| &\leq \left| \sum_n (F_a H^{-1/2}\psi, \bar{F}_n F_a E_n H^{1/2}\psi) \right| \\
&\quad + \left| \sum_n (F_a H^{-1/2}\psi, O(\frac{1}{at})(R2^{-n})^{-1}O(1)E_n H^{1/2}\psi) \right| \\
(3.17) \quad &\leq c \left\| F_a H^{-1/2}\psi \right\| \sum_n \left\| 2^{+n/2} E_n \psi \right\| \frac{1}{t} \frac{R}{a}.
\end{aligned}$$

It follows that,

$$(3.18) \quad (\psi(t), F_a^2 \psi(t)) \leq c \left\| F_a H^{-1/2} \psi_0 \right\| \left\| |\ln H|^{\frac{1+\varepsilon}{2}} H^{-1/2} < x >^{1/2} \psi_0 \right\| o(1) + Q,$$

$$(3.19) \quad Q \equiv \left( H^{-1/2} \psi(t), \left[ H^{1/2}, F_a^2 \right] \psi(t) \right).$$

To control Q, (3.19), we need to commute fractional powers of  $H$ . To this end we use that:

$$H^\alpha = c_\alpha \int_0^\infty \frac{\lambda^{\alpha-1}}{\lambda + H} H d\lambda,$$

and estimate,

$$\begin{aligned}
\lambda^{\alpha-1} \left[ \frac{H}{H+\lambda}, F_a \right] &= \lambda^{\alpha-1} H \left[ \frac{1}{H+\lambda}, F_a \right] + [H, F_a] \frac{\lambda^{\alpha-1}}{H+\lambda} \\
&= \frac{-H}{H+\lambda} O\left(\frac{1}{t} F'_a\right) \frac{\lambda^{\alpha-1}}{H+\lambda} + O\left(\frac{1}{t} F'_a\right) \frac{\lambda^{\alpha-1}}{H+\lambda} \equiv \circledast \\
(3.20) \quad \int_0^\infty d\lambda \circledast &= O(1) O\left(\frac{1}{t} F'_a\right) O(H^{-\alpha}).
\end{aligned}$$

Therefore, using (3.20) with  $\alpha = 1/2$ , we have that

$$|Q| = |(H^{-1/2}\psi(t), O(1)O(\frac{1}{t})H^{-1/2}\psi(t))| \leq \frac{c}{t} \|H^{-1/2}\psi(t)\|^2 \leq \frac{c}{t} \|< x >^{1/2} \psi(0)\|^2. \blacksquare$$

End of Proof.

**REFERENCES**

[BFS] V. Bach, J. Fröhlich, and I.M. Sigal.: Commun. Math. Phys., **207** :249-290, 1999

[BFSS] V. Bach, J. Fröhlich, I. M. Sigal, A. Soffer, *Positive Commutators and Spectrum of Pauli-Fierz Hamiltonians of Atoms and Molecules*, Comm. Math. Phys., **207**, 1999, 557-587.

[FGS] J. Fröhlich, M. Griesemer, I. M. Sigal *Spectral renormalization and Local Decay...*, (2009), arxiv0904.1014v1

[HSS] W. Hunziker, I. M. Sigal, A. Soffer, *Minimal Velocity Bounds*, Comm. PDE, **24**, (1999), No. 11/12, 2279-2295.

[Sig-Sof] I.M. Sigal and A. Soffer, *Local Decay and Propagation Estimates for Time Dependent and Time Independent Hamiltonians*, Preprint, Princeton 1988, (<ftp://www.math.rutgers.edu/pub/soffer>).

MATH DEPT. RUTGERS UNIV. NJ, USA.